Gauge Independence in terms of the Functional Integral*

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Abstract

Among various approaches in proving gauge independence, models containing an explicit gauge dependence are convenient. The well-known example is the gauge parameter in the covariant gauge fixing which is of course most suitable for the perturbation theory but a negative metric prevents us from imaging a dynamical picture. Noncovariant gauge such as the Coulomb gauge is on the contrary used for many physical situations. Therefore it is desirable to include both cases. More than ten years ago, Steinmann introduced a function (distribution) which can play this role in his attempt on discussing quantum electrodynamics (QED) in terms of the gauge invariant fields solely. The method is, however, broken down in the covariant case: the invariant operators are ill-defined because of $1/p^2$ singularity in the Minkowski space. In this paper, we apply his function to the path integral: utilizing the arbitrariness of the function we first restrict it to be able to have a well-defined operator, and then a Hamiltonian with which we can build up the (Euclidean) path integral formula. Although the formula is far from covariant, a full covariant expression is recovered by reviving the components which have been discarded under the construction of the Hamiltonian. There is no pathological defects contrary to the operator formalism. With the aid of the path integral formula, the gauge independence of the free energy as well as the S-matrix is proved. Moreover the reason is clarified why it is so simple and straightforward to argue gauge transformations in the path integral. Discussions on the quark confinement is also presented.

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1 Introduction

Gauge transformations in quantum mechanics reads as the unitary transformation so that proving gauge invariance is nothing but proving the unitary equivalence between two theories[1]. However in quantum field theories, it is well known that canonical commutation relations demand to fix the gauge, that is, each gauge has its own Hilbert space. Consequently in order to assert that the result is gauge independent, we usually compare results which have been obtained by different gauges[2]. (Some approaches, however, treat gauge transformation itself even in quantum field theories[3].) Therefore, models with an explicit gauge dependence may be suitable; for instance α in the Nakanishi-Lautrup formalism[4],

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^{\mu}\partial_{\mu}B + \frac{\alpha}{2}B^2 \ . \tag{1.1}$$

(Throughout the paper, repeated indices are implied summation unless otherwise stated.) Covariance is indispensable in perturbation theories but requires the negative metric which makes it hard to imagine dynamics. On the contrary, noncovariant, such as the Coulomb or the axial gauge can be formulated in an ordinary Hilbert space with a positive metric; which is more useful in physical situations[5]. Therefore it is desirable for a model to include both cases.

More than ten years ago, Steinmann[6] discussed quantum electrodynamics (QED) perturbatively in terms of gauge invariant fields which are defined by the operators, ψ, $\overline{\psi}$, and A^{μ} in the Gupta-Bleuler formalism:

$$\begin{aligned} \Psi(x) &\equiv \exp\left[-ie\int d^4y\phi^{\mu}(x-y)A_{\mu}(y)\right]\psi(x) \ , \\ \overline{\Psi}(x) &\equiv \Psi^{\dagger}\gamma_0 \ , \end{aligned} \tag{1.2}$$

with $\phi^{\mu}(x)$ being real function (distribution strictly speaking) satisfying

$$\partial_{\mu}\phi^{\mu}(x) = \delta^4(x) \ . \tag{1.3}$$

Therefore by noting

$$F^{\mu\nu} \equiv \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \tag{1.4}$$

all the equations of motion become gauge invariant:

$$\partial^{\nu}F_{\mu\nu}(x) = ej_{\mu}(x) \ , \tag{1.5}$$

$$(i\partial\!\!\!/ - m)\Psi(x) = e\gamma^{\mu}\int d^4y\phi^{\nu}(x-y)F_{\mu\nu}(y)\Psi(x) \ , \tag{1.6}$$

It was then shown that all the Wightman functions can be calculated perturbatively. Contrary to his intention that the result must be gauge invariant but that is of course ϕ_{μ}-dependent. Indeed we can regard his method as the ϕ-gauge fixing[7]; which can be seen by noting the following quantity,

$$\langle 0| T^{*}A^{\lambda_1}(x_1;\phi)\cdots A^{\lambda_n}(x_n;\phi)\Psi(y_1)\cdots\Psi(y_m)\overline{\Psi}(z_1)\cdots\overline{\Psi}(z_m)|0\rangle \ , \tag{1.7}$$

with T^* designating the covariant T^*-product, where $A^{\mu}(x;\phi)$ is physical, that is, gauge invariant photon field, given by

$$A_{\mu}(x;\phi) \equiv -\int d^4y\phi^{\nu}(x-y)F_{\mu\nu}(y) \ . \tag{1.8}$$

By a perturbative calculation we can see that the original photon propagator,

$$D^{\mu\nu}(q) \equiv \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_{\mu}q_{\nu}}{q^2}\right) \tag{1.9}$$

must be replaced to the ϕ-dependent one,

$$D^{\mu\nu}(q;\phi) \equiv \frac{-i}{q^2}\Big\{g^{\mu\nu} + iq^{\mu}\widetilde{\phi}^{\nu}(q) - i\widetilde{\phi}^{\mu *}(q)q^{\nu} + q^{\mu}q^{\nu}|\widetilde{\phi}(q)|^2\Big\} \ , \tag{1.10}$$

with $\widetilde{\phi}_\mu(q)$ being the Fourier transform of $\phi_\mu(x)$, in the ordinary n-point function:

$$\langle 0|\, TA^{\lambda_1}(x_1)\cdots A^{\lambda_n}(x_n)\psi(y_1)\cdots\psi(y_m)\overline{\psi}(z_1)\cdots\overline{\psi}(z_m)\,|0\rangle\ . \tag{1.11}$$

Since ϕ_μ is an arbitrary function, the choices,

$$\phi^\mu(x) = \Big(0, \frac{\boldsymbol{\nabla}}{\boldsymbol{\nabla}^2}\delta(x^0)\Big) \equiv \Big(0, \frac{\boldsymbol{x}}{4\pi|\boldsymbol{x}|^3}\delta(x^0)\Big)\ , \tag{1.12}$$

$$\phi^\mu(x) = \Big(0, 0, 0, \delta(x^0)\delta(x^1)\delta(x^2)\theta(x^3)\Big)\ , \tag{1.13}$$

and

$$\phi_\mu(x) = \frac{\partial_\mu}{\Box}\ , \qquad \Big(\widetilde{\phi}_\mu(q) = \frac{iq_\mu}{q^2}\ ,\Big) \tag{1.14}$$

give us the Coulomb, the axial, and the covariant Landau gauge, respectively. However, there is a problem in dividing q^2; which brings $\Psi(x)$ and $\overline{\Psi}(x)$ to ill-defined operators. If we adopt $1/(q^2 - i\epsilon)$ prescription to avoid the singularity, the reality of $\phi_\mu(x)$ is lost. If then we use $1/(q^2 + i\epsilon q_0)$, or $1/(q^2 - i\epsilon q_0)$, or the principal value prescription for preserving the reality, time reversal invariance or causality is broken down. Actually the support of $\phi_\mu(x)$ must be spacelike, since the timelike support causes the difficulty in the ordering between ψ and $\overline{\psi}$ and A_μ. Therefore as far as the operator formalism is concerned, covariance is superficial in the Steinmann's approach.

Contrary to the operator formalism, it is known that the path integral formalism can handle gauge invariance more efficiently. Furthermore, the Euclidean path integral expression, when an imaginary time goes to infinity[8], contains all the information of the ground states of the theory. In this study, we start with the spacelike ϕ_μ so as to throw away the redundant variables and perform canonical quantization without any problem. Then the Euclidean path integral expression is built by the trace formula of an imaginary time evolution operator; which is, of course, far from covariant. At the final stage, the redundant variables are revived by means of insertions of some identities into the path integral[9]. In the formula, we can take any choice of ϕ_μ by means of the Faddeev-Popov trick. In this way, all the above difficulties can be avoided. These are the contents of §2. In §3, the proof for gauge independence of the free energy and the S-matrix is presented. In the next §4, we clarify the reason for ability of discussing the gauge invariance more straightforwardly in the functional method; since contrary to many discussions on gauge transformations using the path integral, there seems to have been no close examination on justification. The final §5 is devoted to discussions.

2 Euclidean Path Integral Expression for an Arbitrary Gauge Fixing

In this section, we construct the Euclidean path integral expression by first reducing the gauge degree of freedom with the aid of ϕ_μ, then applying canonical quantization to this highly noncovariant system. By utilizing the functional representation[1], the (noncovariant) path integral expression is obtained, which finally comes back to a covariant form by reviving the redundant degrees in terms of insertions of some identities.

Assume that the support of $\phi_\mu(x)$ are spacelike:

$$\phi_\mu(x) = (0, f_i(\boldsymbol{x})\delta(x_0))\ . \tag{2.1}$$

In the momentum space,

$$\widetilde{\phi}_\mu(p) = (0, \widetilde{f}_i(\boldsymbol{p}))\ , \tag{2.2}$$

with

$$\phi_\mu(x) = \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{ipx}\widetilde{\phi}_\mu(p)\ , \tag{2.3}$$

then the reality of $\phi_\mu(x)$ and the relation (1.3) turn out to be

$$\widetilde{\phi}^*_\mu(p) = \widetilde{\phi}_\mu(-p)\ , \qquad p^\mu\widetilde{\phi}_\mu(p) = i\ , \tag{2.4}$$

[1]The path integral formula by means of the holonomic representation[10] always suffers from the nonlocal bilinear term in relativistic cases due to anti-particle contributions. Therefore it is unsuitable to handle with gauge transformations.

respectively. With the help of ϕ_μ's, we can extract the gauge degree of freedom from A^μ such that

$$A^\mu(x) = A^\mu(x;\phi) + \partial^\mu \omega(x) \ , \tag{2.5}$$

where

$$\begin{aligned} A^\mu(x;\phi) &\equiv \int d^4y \{\delta^\mu_\nu \delta^4(x-y) - \partial^\mu_x \phi_\nu(x-y)\} A^\nu(y) \ , & (2.6) \\ \omega(x) &\equiv \int d^4y \phi^\mu(x-y) A_\mu(y) \ . & (2.7) \end{aligned}$$

The relation (2.6) is nothing but (1.8) with the use of (1.4) as well as (1.3) . In order to discuss path integral the Schrödinger picture is employed so that the time argument of the fields is omitted, thus $A^i(\boldsymbol{x})$ is used for the time being. Therefore the relations, (2.5) to (2.7), read as

$$\begin{aligned} A^i(\boldsymbol{x}) &= A^i(\boldsymbol{x};\phi) + \boldsymbol{\nabla}^i \omega(\boldsymbol{x}) \ , & (2.8) \\ A^i(\boldsymbol{x};\phi) &= \int d^3y \{\delta^{ij}\delta^3(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{\nabla}^i_x f^j(\boldsymbol{x}-\boldsymbol{y})\} A^j(\boldsymbol{y}) \ , & (2.9) \\ \omega(\boldsymbol{x}) &= \int d^3y f^j(\boldsymbol{x}-\boldsymbol{y}) A^j(\boldsymbol{y}) \ . & (2.10) \end{aligned}$$

Since a genuine physical component must have two components, we must select them out from three $A^i(\boldsymbol{x})$'s: to this end consider the norm of functional space of $A^i(\boldsymbol{x})$:

$$\begin{aligned} \int d^3x \ \delta A^i(\boldsymbol{x}) \ \delta A^i(\boldsymbol{x}) &= \int d^3x \{\delta A^i(\boldsymbol{x};\phi)\delta A^i(\boldsymbol{x};\phi) - \delta\omega(\boldsymbol{x}) \boldsymbol{\nabla}^i \delta A^i(\boldsymbol{x};\phi) \\ &\quad + \delta A^i(\boldsymbol{x};\phi) \boldsymbol{\nabla}^i \delta\omega(\boldsymbol{x}) - \delta\omega(\boldsymbol{x}) \boldsymbol{\nabla}^2 \delta\omega(\boldsymbol{x})\} \ , \end{aligned} \tag{2.11}$$

whose $\delta A^i(x;\phi) \ \delta A^i(x;\phi)$ part reads

$$\int d^3x \ \delta A^i(\boldsymbol{x};\phi) \ \delta A^i(\boldsymbol{x};\phi) = \int \frac{d^3p}{(2\pi)^3} \delta \widetilde{A}^{j*}(\boldsymbol{p}) M^{jk}(\boldsymbol{p}) \delta \widetilde{A}^k(\boldsymbol{p}) \ , \tag{2.12}$$

with $\widetilde{A}^k(\boldsymbol{p})$ satisfying $\widetilde{A}^k(-\boldsymbol{p}) = \widetilde{A}^{k*}(\boldsymbol{p})$. Here

$$M^{jk}(\boldsymbol{p}) \equiv \delta^{jk} + i\widetilde{f}^{j*}(\boldsymbol{p})p^k - ip^j \widetilde{f}^k(\boldsymbol{p}) + \boldsymbol{p}^2 |\widetilde{\boldsymbol{f}}(\boldsymbol{p})|^2 \tag{2.13}$$

which can be diagonalized as[11]

$$n^j_{(\alpha)}(\boldsymbol{p}) M^{jk}(\boldsymbol{p}) n^{k*}_{(\beta)}(\boldsymbol{p}) = \begin{pmatrix} 1 & & \\ & \boldsymbol{p}^2|\widetilde{\boldsymbol{f}}(\boldsymbol{p})|^2 & \\ & & 0 \end{pmatrix}_{\alpha\beta} \ , \tag{2.14}$$

where

$$\begin{aligned} n^k_{(1)}(\boldsymbol{p}) &\equiv \epsilon^{klm} n^l_{(2)}(\boldsymbol{p}) n^m_{(3)}(\boldsymbol{p}) \ , \\ n^k_{(2)}(\boldsymbol{p}) &\equiv \{ip^k + \boldsymbol{p}^2 \widetilde{f}^k(\boldsymbol{p})\} \Big/ \sqrt{\boldsymbol{p}^2(\boldsymbol{p}^2|\widetilde{\boldsymbol{f}}(\boldsymbol{p})|^2 - 1)} \ , & (2.15) \\ n^k_{(3)}(\boldsymbol{p}) &\equiv ip^k/|\boldsymbol{p}| \ , \end{aligned}$$

are the orthonormal base obeying

$$\sum_{\alpha=1}^{3} n^{j*}_{(\alpha)}(\boldsymbol{p}) n^k_{(\alpha)}(\boldsymbol{p}) = \delta^{jk}, \qquad n^{k*}_{(\alpha)}(\boldsymbol{p}) = n^k_{(\alpha)}(-\boldsymbol{p}) \ . \tag{2.16}$$

In view of (2.14) we can take the desired physical components as $\widetilde{A}^{(1)}(\boldsymbol{p})$ and $\widetilde{A}^{(2)}(\boldsymbol{p})$ where

$$\widetilde{A}^{(\alpha)}(\boldsymbol{p}) \equiv n^k_{(\alpha)}(\boldsymbol{p}) \widetilde{A}^k(\boldsymbol{p}); \qquad (\alpha = 1, 2) \ . \tag{2.17}$$

Therefore according to (2.9) $A^i(\boldsymbol{x};\phi)$ is now expressed solely by the physical components:

$$\begin{aligned} A^i(\boldsymbol{x};\phi) &= \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \{ n^{i*}_{(1)}(\boldsymbol{p}) \widetilde{A}^{(1)}(\boldsymbol{p}) + [\delta^{ij} - ip^i \widetilde{f}^j(\boldsymbol{p})] n^{j*}_{(2)}(\boldsymbol{p}) \widetilde{A}^{(2)}(\boldsymbol{p}) \} \\ &= n^{i*}_{(1)}(-i\boldsymbol{\nabla}) A^{(1)}(\boldsymbol{x}) + [\delta^{ij} - \boldsymbol{\nabla}^i \widetilde{f}^j(-i\boldsymbol{\nabla})] n^{j*}_{(2)}(-i\boldsymbol{\nabla}) A^{(2)}(\boldsymbol{x}) \ , \end{aligned} \tag{2.18}$$

where as usual

$$A^{(\alpha)}(\boldsymbol{x}) \equiv \int \frac{d^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \widetilde{A}^{(\alpha)}(\boldsymbol{p}) \ ; \qquad (\alpha = 1, 2) \ . \tag{2.19}$$

Finally the norm (2.11) is given by

$$\begin{aligned} &\int d^3x \delta A^i(\boldsymbol{x}) \delta A^i(\boldsymbol{x}) \\ &= \int d^3x \begin{pmatrix} \delta A^{(1)} & \delta A^{(2)} & \delta\omega \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 & -\dfrac{\sqrt{\quad\bullet\quad}}{-\boldsymbol{\nabla}^2} \\ 0 & -\sqrt{\quad\bullet\quad} & -\boldsymbol{\nabla}^2 \end{pmatrix} \begin{pmatrix} \delta A^{(1)} \\ \delta A^{(2)} \\ \delta\omega \end{pmatrix} \ , \end{aligned} \tag{2.20}$$

where

$$\sqrt{\quad\bullet\quad} \equiv \sqrt{\boldsymbol{\nabla}^2 (\boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 + 1)} \ . \tag{2.21}$$

The action, with the source term J^μ, reads

$$\begin{aligned} S &= \int d^4x \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\nu \right\} \\ &= \int d^4x \left\{ \frac{1}{2} \sum_{i,j=1}^{3} A^i(x;\phi)(\delta^{ij}\boldsymbol{\nabla}^2 - \boldsymbol{\nabla}^i \boldsymbol{\nabla}^j) A^j(x;\phi) + \frac{1}{2} \dot{\boldsymbol{A}}^2(x;\phi) - A^0(x;\phi) \boldsymbol{\nabla}\cdot\dot{\boldsymbol{A}}(x;\phi) \right. \\ &\quad \left. + \frac{1}{2} (\boldsymbol{\nabla} A^0(x;\phi))^2 + J_0(x) A^0(x;\phi) - \boldsymbol{J}(x)\cdot\boldsymbol{A}(x;\phi) \right\} \ , \end{aligned} \tag{2.22}$$

which is further rewritten, with the help of (2.18) as well as (2.15), as

$$\begin{aligned} S = \int d^4x &\left\{ \frac{1}{2} \sum_{\alpha=1}^{2} A^{(\alpha)}(x) \boldsymbol{\nabla}^2 A^{(\alpha)}(x) + \frac{1}{2} \left(\dot{A}^{(1)}(x) \right)^2 - \frac{1}{2} \dot{A}^{(2)}(x) \boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 \dot{A}^{(2)}(x) \right. \\ &\left. -A^0(x;\phi) \sqrt{\boldsymbol{\nabla}^2 (\boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 + 1)} \dot{A}^{(2)}(x) + \frac{1}{2} (\boldsymbol{\nabla} A^0(x;\phi))^2 + J_0(x) A^0(x;\phi) - \boldsymbol{J}(x)\cdot \Big(\boldsymbol{A}(x;\phi) \Big) \right\} \ , \end{aligned} \tag{2.23}$$

whose last term, $\big(\boldsymbol{A}(x;\phi)\big)$, implies that $\boldsymbol{A}(x;\phi)$ has been given by $A^{(1)}$ and $A^{(2)}$ by the relation (2.18). The system still remains in a constrained system because $A^0(x;\phi)$ is not a dynamical variable. Solving the constraint, we obtain the Hamiltonian:

$$\begin{aligned} H(t) &= \int d^3x \left\{ \frac{1}{2} \sum_{\alpha=1}^{2} \left[\Big(\Pi^{(\alpha)}(\boldsymbol{x}) \Big)^2 + \Big(\boldsymbol{\nabla} A^{(\alpha)}(\boldsymbol{x}) \Big)^2 \right] \right. \\ &\quad \left. + J_0(x) \frac{\sqrt{\boldsymbol{\nabla}^2 (\boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 + 1)}}{\boldsymbol{\nabla}^2} \Pi^{(2)}(\boldsymbol{x}) + \frac{1}{2} J_0(x) |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 J_0(x) + \boldsymbol{J}(x)\cdot(\boldsymbol{A}(\boldsymbol{x};\phi)) \right\} \ , \end{aligned} \tag{2.24}$$

where $x \equiv (t, \boldsymbol{x})$ and $\Pi^{(\alpha)}$ is the canonical conjugate momentum of $A^{(\alpha)}$. Note that the explicit time dependence lies in the sources so as to indicate the Hamiltonian.

Now quantization can be carried out by means of the canonical commutation relations:

$$[\hat{A}^{(\alpha)}(\boldsymbol{x}), \hat{\Pi}^{(\beta)}(\boldsymbol{y})] = i\delta^{\alpha\beta} \delta^3(\boldsymbol{x} - \boldsymbol{y}) \ , \quad [\hat{A}^{(\alpha)}(\boldsymbol{x}), \hat{A}^{(\beta)}(\boldsymbol{y})] = [\hat{\Pi}^{(\alpha)}(\boldsymbol{x}), \hat{\Pi}^{(\beta)}(\boldsymbol{y})] = 0 \ , \tag{2.25}$$

where caret denotes an operator. The partition function in our interest is defined by

$$Z_T[J] \equiv \lim_{N\to\infty} \mathrm{Tr}\left[(\mathbf{I} - \Delta\tau \hat{H}_N)(\mathbf{I} - \Delta\tau \hat{H}_{N-1}) \cdots (\mathbf{I} - \Delta\tau \hat{H}_1)\right] \ , \tag{2.26}$$

where $\Delta\tau \equiv T/N$, $H_j \equiv H(j\Delta\tau)$, and the source, $J_\mu(x)$, has been assumed to be analytically continuable. Here Tr can be taken for any complete set, but a functional representation,

$$\hat{A}^{(\alpha)}(\boldsymbol{x})\,|\{A\}\rangle = A^{(\alpha)}(\boldsymbol{x})\,|\{A\}\rangle \,, \qquad \hat{\Pi}^{(\alpha)}(\boldsymbol{x})\,|\{\Pi\}\rangle = \Pi^{(\alpha)}(\boldsymbol{x})\,|\{\Pi\}\rangle \ , \tag{2.27}$$

with completeness,

$$\int \mathcal{D}A^{(\alpha)}\,|\{A\}\rangle\,\langle\{A\}| = \mathbf{I} \ , \qquad \int \mathcal{D}\Pi^{(\alpha)}\,|\{\Pi\}\rangle\,\langle\{\Pi\}| = \mathbf{I} \ , \tag{2.28}$$

should be chosen so as to obtain the Euclidean path integral representation[9]:

$$\begin{aligned} Z_T[J] &= \lim_{N\to\infty} \mathcal{N}^{2N} \int \prod_{k=1}^{N} \mathcal{D}A_k^{(\alpha)} \mathcal{D}\Pi_k^{(\alpha)} \\ &\quad \times \exp\left[\Delta\tau \sum_{k=1}^{N} \left\{ \int d^3x\, i \sum_{\alpha=1}^{2} \Pi_k^{(\alpha)}(\boldsymbol{x}) \frac{A_k^{(\alpha)}(\boldsymbol{x}) - A_{k-1}^{(\alpha)}(\boldsymbol{x})}{\Delta\tau} - H_k(\Pi_k^{(\alpha)}, A_k^{(\alpha)}) \right\}\right] \ , \end{aligned} \tag{2.29}$$

where $\mathcal{N}$ is the normalization factor defined in

$$\langle\{\Pi\}|\{A\}\rangle \equiv \mathcal{N} \exp\left[-i \int d^3x \sum_{\alpha=1}^{2} \Pi^{(\alpha)}(\boldsymbol{x}) A^{(\alpha)}(\boldsymbol{x})\right] \ . \tag{2.30}$$

More explicitly

$$\mathcal{N} = \prod_{\boldsymbol{x}} \frac{1}{2\pi} \ . \tag{2.31}$$

Due to the trace, the boundary condition is periodic $A_0^{(\alpha)}(\boldsymbol{x}) = A_N^{(\alpha)}(\boldsymbol{x})$. (This is a formal expression and is ill-defined actually. To make $Z_T[J]$ well-defined, it is also necessary to discretize the space part and to put the system in a box. For gauge theories, there still remains the difficulty of the infrared divergence which can, however, be avoided by making further considerations[9]. Hereafter we confine ourselves in a continuous representation for notational simplicity.)

Therefore we write

$$\begin{aligned} Z_T[J] &= \int \mathcal{D}A^{(\alpha)} \mathcal{D}\Pi^{(\alpha)} \exp\left[\int d^4x_{_E} \left\{ i \sum_{\alpha=1}^{2} \Pi^{(\alpha)}\ \dot{A}^{(\alpha)}\ - H(\Pi, A) \right\}\right] \\ &= \int \mathcal{D}A^{(\alpha)} \mathcal{D}\Pi^{(\alpha)} \exp\Bigg[\int d^4x_{_E} \Bigg\{ i \sum_{\alpha=1}^{2} \Pi^{(\alpha)}(\tau, \boldsymbol{x})\ \dot{A}^{(\alpha)}(\tau, \boldsymbol{x}) \\ &\quad - \frac{1}{2} \sum_{\alpha=1}^{2} \left[\left(\Pi^{(\alpha)}(\tau, \boldsymbol{x})\right)^2 + \left(A^{(\alpha)}(\tau, \boldsymbol{x})\right)^2 \right] + iJ_4(\tau, \boldsymbol{x}) \frac{\sqrt{\boldsymbol{\nabla}^2(\boldsymbol{\nabla}^2 |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 + 1)}}{\boldsymbol{\nabla}^2} \Pi^{(2)}(\tau, \boldsymbol{x}) \\ &\quad + \frac{1}{2} J_4(\tau, \boldsymbol{x}) |\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2 J_4(\tau, \boldsymbol{x}) - \boldsymbol{J}(\tau, \boldsymbol{x}) \cdot (\boldsymbol{A}(\tau, \boldsymbol{x}; \phi)) \Bigg\}\Bigg], \end{aligned} \tag{2.32}$$

where

$$\int d^4x_{_E} \equiv \int_0^T d\tau \int d^3x \ , \tag{2.33}$$

$$J_4 \equiv iJ_0 \ . \tag{2.34}$$

Integrating with respect to $\Pi^{(\alpha)}$, then inserting the Gaussian identity,

$$\mathbf{I} = \int \mathcal{D}A^4(\tau, \boldsymbol{x}) [\det(-\boldsymbol{\nabla}^2)]^{\frac{1}{2}} \exp\left[-\int d^4x_{_E} \frac{1}{2} A^4(\tau, \boldsymbol{x})(-\boldsymbol{\nabla}^2) A^4(\tau, \boldsymbol{x})\right] \ , \tag{2.35}$$

and introducing the new integration variable such that

$$A^4(\tau,\boldsymbol{x};\phi) \equiv A^4(\tau,\boldsymbol{x}) + \frac{\sqrt{\boldsymbol{\nabla}^2(\boldsymbol{\nabla}^2|\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2+1)}}{\boldsymbol{\nabla}^2}\,\dot{A}^{(2)}(\tau,\boldsymbol{x}) + \frac{1}{\boldsymbol{\nabla}^2}J_4(\tau,\boldsymbol{x})\ , \tag{2.36}$$

we obtain

$$\begin{aligned} Z_T[J] = &\int\mathcal{D}A^{(\alpha)}\mathcal{D}A^4(\tau,\boldsymbol{x};\phi)[\det(-\boldsymbol{\nabla}^2)]^{\frac{1}{2}} \\ &\times\exp\Bigg[-\int d^4x_{\scriptscriptstyle E}\Bigg\{\frac{1}{2}\Big(\dot{A}^{(1)}(\tau,\boldsymbol{x})\Big)^2 - \frac{1}{2}\,\dot{A}^{(2)}(\tau,\boldsymbol{x})\boldsymbol{\nabla}^2|\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2\,\dot{A}^{(2)}(\tau,\boldsymbol{x}) \\ &-\frac{1}{2}\sum_{\alpha=1}^{2}A^{(\alpha)}(\tau,\boldsymbol{x})\boldsymbol{\nabla}^2A^{(\alpha)}(\tau,\boldsymbol{x}) - \frac{1}{2}A^4(\tau,\boldsymbol{x};\phi)\boldsymbol{\nabla}^2A^4(\tau,\boldsymbol{x};\phi) \\ &+A^4(\tau,\boldsymbol{x};\phi)\sqrt{\boldsymbol{\nabla}^2(\boldsymbol{\nabla}^2|\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})|^2+1)}\,\dot{A}^{(2)}(\tau,\boldsymbol{x}) + J_4(\tau,\boldsymbol{x})A^4(\tau,\boldsymbol{x};\phi) + \boldsymbol{J}(\tau,\boldsymbol{x})\cdot(\boldsymbol{A}(\tau,\boldsymbol{x};\phi))\Bigg\}\Bigg]\ . \end{aligned} \tag{2.37}$$

In this way $A^4(\tau,\boldsymbol{x};\phi)$ has been recovered. Now by recalling the relations (2.22) and (2.23), this becomes

$$\begin{aligned} Z_T[J] \;=\; &\int\mathcal{D}A^{(\alpha)}\mathcal{D}A^4(\tau,\boldsymbol{x};\phi)[\det(-\boldsymbol{\nabla}^2)]^{\frac{1}{2}} \\ &\times\exp\Big[-\int d^4x_{\scriptscriptstyle E}\Big\{\frac{1}{4}F_{\mu\nu}(\tau,\boldsymbol{x};\phi)F_{\mu\nu}(\tau,\boldsymbol{x};\phi) + J_\mu(\tau,\boldsymbol{x})A_\mu(\tau,\boldsymbol{x};\phi)\Big\}\Big]\ , \end{aligned} \tag{2.38}$$

where

$$F_{\mu\nu}(\tau,\boldsymbol{x};\phi) \equiv \partial_\mu A_\nu(\tau,\boldsymbol{x};\phi) - \partial_\nu A_\mu(\tau,\boldsymbol{x};\phi)\ , \tag{2.39}$$

and $\mu,\nu = 1,2,3,4$. $A^i(\tau,\boldsymbol{x};\phi)$ is now defined, in view of (2.18), by

$$A^i(\tau,\boldsymbol{x};\phi) \equiv n^{i*}_{(1)}(-i\boldsymbol{\nabla})A^{(1)}(\tau,\boldsymbol{x}) + [\delta^{ij} - \boldsymbol{\nabla}^i\widetilde{f}^j(-i\boldsymbol{\nabla})]n^{j*}_{(2)}(-i\boldsymbol{\nabla})A^{(2)}(\tau,\boldsymbol{x})\ . \tag{2.40}$$

In (2.38), almost everything is recovered but the functional measure which still consists of three components. To cure this, the gauge degree of freedom ω, (2.10),

$$\omega(\tau,\boldsymbol{x}) = \int d^3y\,\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{A}(\tau,\boldsymbol{y}) = \widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})\cdot\boldsymbol{A}(\tau,\boldsymbol{x})\ , \tag{2.41}$$

is revived, by means of the delta function, giving

$$Z_T[J] = \int\mathcal{D}A_\mu\delta(\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla})\cdot\boldsymbol{A}(\tau,\boldsymbol{x}))\exp\left[-\int d^4x_{\scriptscriptstyle E}\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu\right\}\right]\ , \tag{2.42}$$

where use has been made of the relation of functional measure,

$$\mathcal{D}A^i = \mathcal{D}A^{(\alpha)}\mathcal{D}\omega[\det(-\boldsymbol{\nabla}^2)]^{\frac{1}{2}}\ , \tag{2.43}$$

obtained from (2.20).

Going back the original notation we find the covariant expression (2.1)

$$Z_T[J] = \int\mathcal{D}A_\mu\delta\Big(\int d^4y_{\scriptscriptstyle E}\phi_\mu(x-y)A_\mu(y)\Big)\exp\left[-\int d^4x_{\scriptscriptstyle E}\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu\right\}\right]\ . \tag{2.44}$$

It is a straightforward task to check that the propagator is correctly given by (1.10).

In the Coulomb case, (1.12), that is, $\widetilde{\boldsymbol{f}}(-i\boldsymbol{\nabla}) \equiv \boldsymbol{\nabla}/\boldsymbol{\nabla}^2$ in (2.42), a familiar expression,

$$Z_T^{\rm coul}[J] = \int\mathcal{D}A_\mu\delta(\boldsymbol{\nabla}\cdot\boldsymbol{A})\left|\det\Big(-\boldsymbol{\nabla}^2\Big)\right|\exp\left[-\int d^4x_{\scriptscriptstyle E}\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu\right\}\right]\ , \tag{2.45}$$

is obtained. Furthermore the troubles in Steinmann's approach are evaded; since the expression has no singularity at all even in the Landau gauge,

$$\phi_\mu(x) = \frac{\partial_\mu}{\Box_E} \ , \quad \Box_E \equiv \partial_\mu\partial_\mu \ , \tag{2.46}$$

and

$$Z_T^{\text{land}}[J] = \int \mathcal{D}A_\mu \delta(\partial_\mu A_\mu) \left|\det\left(-\Box_E\right)\right| \exp\left[-\int d^4x_E \left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + J_\mu A_\mu\right\}\right] \ . \tag{2.47}$$

3 Proof of Gauge Independence

In this section, gauge independence of (2.44) is proved; in other words, we show that any choice of ϕ^μ leads to the same result in the case of the free energy as well as the S-matrix. To this end, let us first study how a gauge transformation affects the expression (2.44): the gauge transformation from A_μ to A'_μ is given by

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \partial_\mu\theta(x) \ . \tag{3.1}$$

The gauge conditions are supposed as

$$\begin{aligned} \int d^4y_E \phi_\mu(x-y)A_\mu(y) &= 0 \ , \\ \int d^4y_E \phi'_\mu(x-y)A'_\mu(y) &= 0 \ , \end{aligned} \tag{3.2}$$

respectively. The second relation can be rewritten as

$$0 = \int d^4y_E \phi'_\mu(x-y)A'_\mu(y) = \int d^4y_E \phi_\mu(x-y)A_\mu(y) + \int d^4y_E \varDelta\phi_\mu(x-y)A_\mu(y) + \theta(x) \ , \tag{3.3}$$

by use of (3.1) and (1.3), where

$$\varDelta\phi_\mu(x) \equiv \phi'_\mu(x) - \phi_\mu(x) \ . \tag{3.4}$$

Therefore under the gauge conditions (3.2), $\theta(x)$ is obtained as

$$\theta(x) = -\int d^4y_E \varDelta\phi_\mu(x-y)A_\mu(y) \ . \tag{3.5}$$

The partition function of QED, by adding the fermionic part to (2.44), is found as

$$\begin{aligned} Z[J,\overline{\eta},\eta] &= \int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\overline{\psi}\delta\left(\int d^4y_E \phi_\mu(x-y)A_\mu(y)\right) \\ &\quad \times \exp\left[-\int d^4x_E \left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{D}+m)\psi + J_\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta\right\}\right] \ , \end{aligned} \tag{3.6}$$

where the periodic boundary condition for ϕ_μ and the anti-periodic boundary condition for fermions must be understood[2]. Meanwhile the transformed partition function is

$$\begin{aligned} Z'[J,\overline{\eta},\eta] &= \int \mathcal{D}A'_\mu \mathcal{D}\psi'\mathcal{D}\overline{\psi}'\delta\left(\int d^4y_E \phi'_\mu(x-y)A'_\mu(y)\right) \\ &\quad \times \exp\left[-\int d^4x_E \left\{\frac{1}{4}F'_{\mu\nu}F'_{\mu\nu} + \overline{\psi}'(\not{D}'+m)\psi' + J_\mu A'_\mu + \overline{\eta}\psi' + \overline{\psi}'\eta\right\}\right] \ , \end{aligned} \tag{3.7}$$

where A'_μ has been given in (3.1) and

$$\psi(x) \mapsto \psi'(x) \equiv e^{i\theta(x)}\psi(x) \ , \quad \overline{\psi}(x) \mapsto \overline{\psi'}(x) \equiv \overline{\psi}(x)e^{-i\theta(x)} \ . \tag{3.8}$$

[2]The continuum representation for fermion is problematic[12]. But here we concentrate ourselves on perturbation theories so that we neglect the Wilson term etc. .

Then a simple change of variables with a trivial Jacobian[3] leads to

$$
\begin{aligned}
Z'[J,\overline{\eta},\eta] &= \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \delta\left(\int d^4y_{{}_E} \phi_\mu(x-y)A_\mu(y)\right) \\
&\quad \times \exp\left[-\int d^4x_{{}_E}\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{D}+m)\psi + J_\mu\left(A_\mu + \partial_\mu\theta\right) + \overline{\eta}\mathrm{e}^{ie\theta}\psi + \overline{\psi}\mathrm{e}^{-ie\theta}\eta\right\}\right] \ . \quad (3.9)
\end{aligned}
$$

Therefore if the sources, $J, \overline{\eta}$, and η, are absent, the relation

$$
Z' = Z\left(= \mathrm{T}r e^{-TH}\right) \ , \quad (3.10)
$$

implies that the free energy of QED is gauge independent. Moreover, it can be recognized that expectation values of a gauge invariant operator, such as the Belinfante's energy-momentum tensor, $\langle\{n\}|\,\Theta_{\mu\nu}(x)\,|\{n\}\rangle$ is gauge invariant[4], where $|\{n\}\rangle$ designates states of electrons and photons, and

$$
\begin{aligned}
\Theta_{\mu\nu} &\equiv \frac{i}{4}\overline{\psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\psi - F_{\mu\rho}F_\nu{}^\rho - g_{\mu\nu}\mathcal{L} \ , \\
\mathcal{L} &\equiv \overline{\psi}\left(\frac{i}{2}\overleftrightarrow{\not{D}} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},
\end{aligned} \quad (3.11)
$$

with

$$
\overline{\psi}\overleftrightarrow{D}_\mu\psi \equiv \overline{\psi}\Big((\partial_\mu - ieA_\mu)\psi\Big) - \Big((\partial_\mu + ieA_\mu)\overline{\psi}\Big)\psi. \quad (3.12)
$$

(We have employed the Minkowski metric, here.) In this way, the path integral gives us a more quick and intuitive derivation of gauge independence, whose reason is clarified in the next section.

In order to discuss a gauge independence of the S-matrix, however, we need a further consideration: suppose $\theta(x)$ is infinitesimal so that the difference between $Z'[J,\overline{\eta},\eta]$ and $Z[J,\overline{\eta},\eta]$ is

$$
\begin{aligned}
\Delta Z[J,\overline{\eta},\eta] &= Z'[J,\overline{\eta},\eta] - Z[J,\overline{\eta},\eta] \\
&= \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \delta\left(\int d^4y_{{}_E} \phi_\mu(x-y)A_\mu(y)\right) \int d^4x_{{}_E}\left[\theta\partial_\mu J_\mu + ie\theta\left(\overline{\psi}\eta - \overline{\eta}\psi\right)\right] \\
&\quad \times \exp\left[-\int d^4x_{{}_E}\left\{\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \overline{\psi}(\not{D}+m)\psi + J_\mu A_\mu + \overline{\eta}\psi + \overline{\psi}\eta\right\}\right] \quad (3.13)
\end{aligned}
$$

which is the generating functional of the Green's function

$$
\Delta G^{(n)} \equiv G'^{(n)} - G^{(n)} \ , \quad (3.14)
$$

with $G'^{(n)}$ and $G^{(n)}$ being the n-point Green's functions in each gauge.

The S-matrix is now found, after rotating back to the Minkowski space, by cutting external legs and multiplying the wave functions of electrons and photons, that is, multiplying

$$
\begin{aligned}
&\frac{\not{p}-m}{i\sqrt{z_2}}\frac{u(\boldsymbol{p},s)}{\sqrt{(2\pi)^3 2p_0}} \ , \ \frac{\overline{u}(\boldsymbol{p},s)}{\sqrt{(2\pi)^3 2p_0}}\frac{\not{p}-m}{i\sqrt{z_2}} \ , \quad \text{for electrons} \\
&\frac{-q^2}{i\sqrt{z_3}}\frac{\xi_\mu^{(i)}(\boldsymbol{q})}{\sqrt{(2\pi)^3 2q_0}} \ , \quad \text{for photons}
\end{aligned} \quad (3.15)
$$

---

[3] Since the Jacobian from A'_μ to A_μ reads $\left|\frac{\delta A'_\mu}{\delta A_\nu}\right| = |\det(\delta_{\mu\nu} - \partial_\mu \Delta\phi_\nu)|$, according to (3.1) and (3.5), which is unity: consider the determinant of the matrix, $M_{ij} = \delta_{ij} + A_iB_j$ with $\sum_i A_iB_i = 0$ to find $\det M = \exp[\mathrm{T}r\log(1+AB)] = 1$; since $\mathrm{T}r(AB)^n = 0$; ${}^\forall n$.

[4] (3.6) itself is gauge invariant without sources for an arbitrary T, that is, $\mathrm{Tr}\Theta_{\mu\nu}(x)e^{-TH} = \sum_{\{n\}}\langle\{n\}|\,\Theta_{\mu\nu}(x)\,|\{n\}\rangle$ is gauge invariant. Then, $T \to \infty$ picks up the expectation value between the vacuum $|0\rangle$; which is gauge invariant. Next, put $T \to \infty$ in the quantity, $Z[J=0,\overline{\eta}=0,\eta=0] - \langle 0|\,\Theta_{\mu\nu}(x)\,|0\rangle$, to give the expectation value between the first excited state; which is again gauge invariant. Repeating the procedures, we find the expectation value between any state is gauge invariant. q.e.d.

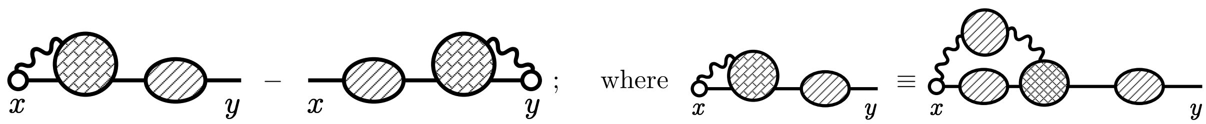

Figure 1: The two-point function in (3.18): the circle denotes the θ-insertion. The blob in the left graphs is collections of the full propagators and the vertex, seen in the right graph.

to the Green's function[5]. Here the photon polarization $\xi_\mu^{(i)}(\boldsymbol{q})$ fulfills the transversal condition $q^\mu \xi_\mu^{(i)}(\boldsymbol{q}) = 0$. Due to this, $\theta\partial_\mu J_\mu$ term in (3.13) (now $\theta\partial^\mu J_\mu$) drops out so that it is enough to concentrate on differences of the electron legs: the gauge dependent part of the S-matrix, S_g, is thus read

$$S_g = \prod_{j=1}^{n} \frac{\overline{u}(\boldsymbol{p}_j, s'_j)}{\sqrt{(2\pi)^3 2(p_j)_0}} \frac{\not{p}_j - m}{i\sqrt{z_2}} G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) \prod_{j=1}^{n} \frac{\not{k}_j - m}{i\sqrt{z_2}} \frac{u(\boldsymbol{k}_j, s_j)}{\sqrt{(2\pi)^3 2(k_j)_0}} \ . \tag{3.16}$$

Consider the Fourier transform of the difference of $2n$-point function:

$$\begin{aligned}
&\Delta G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n)\Big|_{k_n = p_n + \sum_{j=1}^{n}(p_j - k_j)} \\
&\equiv \int \prod_{j=1}^{n-1} (d^4x_j d^4y_j) d^4x_n \exp\left[\sum_{j=1}^{n-1}(ip_jx_j - ik_jy_j) + ip_nx_n\right] \\
&\qquad \times \frac{\delta^{2n}}{\delta\overline{\eta}(x_1)\cdots\delta\overline{\eta}(x_n)\delta\eta(y_1)\cdots\delta\eta(y_{n-1})\delta\eta(0)} \Delta Z[J, \overline{\eta}, \eta]\Big|_{J=\overline{\eta}=\eta=0} \\
&= \int \prod_{j=1}^{n-1} (d^4x_j d^4y_j) d^4x_n \exp\left[\sum_{j=1}^{n-1}(ip_jx_j - ik_jy_j) + ip_nx_n\right] \\
&\quad \times ie \langle 0| \mathrm{T}\left\{\sum_{j=1}^{n-1}[\theta(x_j) - \theta(y_j)] + [\theta(x_n) - \theta(0)]\right\} \psi(x_1)\cdots\psi(x_n)\overline{\psi}(y_1)\cdots\overline{\psi}(y_{n-1})\overline{\psi}(0) |0\rangle \ .
\end{aligned} \tag{3.17}$$

In view of (3.5), θ contains A_μ and cannot be put outside of the expectation value. The two-point function, that is, $n = 1$ case,

$$\Delta G^{(2)}(x, y) = ie \langle 0| \mathrm{T}(\theta(x) - \theta(y))\psi(x)\overline{\psi}(y) |0\rangle \ , \tag{3.18}$$

depicted in figure 1,

becomes near the mass-shell such that

$$\Delta G^{(2)}(p) \sim \overline{A}(p)\Big|_{\not{p}=m} \frac{iz_2}{\not{p} - m} - \frac{iz_2}{\not{p} - m} A(p)\Big|_{\not{p}=m} \ , \tag{3.19}$$

where

$$\begin{aligned}
\overline{A}(p) &\overset{F.T.}{=} \int d^4z \ ie \langle 0| \mathrm{T}\theta(x)\psi(x)\overline{\psi}(z) |0\rangle (G^{(2)})^{-1}(z, y) \ , \\
A(p) &\overset{F.T.}{=} \int d^4z \ ie (G^{(2)})^{-1}(x, z) \langle 0| \mathrm{T}\theta(y)\psi(z)\overline{\psi}(y) |0\rangle \ ,
\end{aligned} \tag{3.20}$$

and $F.T.$ designates the Fourier transformation; since the electron two-point function behaves

$$G^{(2)}(p) \sim \frac{iz_2}{\not{p} - m} \ , \tag{3.21}$$

[5]It is troublesome to write out the LSZ-asymptotic state for electrons in this way; since in a noncovariant gauge z_2 is given as matrix-valued acting differently on each spinor index. However, there are additional renormalization conditions, since the self-energy is not merely the function of $\not{p}$: it depends on $p_0\gamma_0$ as well as $p_k\gamma_k$ in the Coulomb gauge for instance. Here we assume that z_2 has already been diagonalized by utilizing these additional degrees.

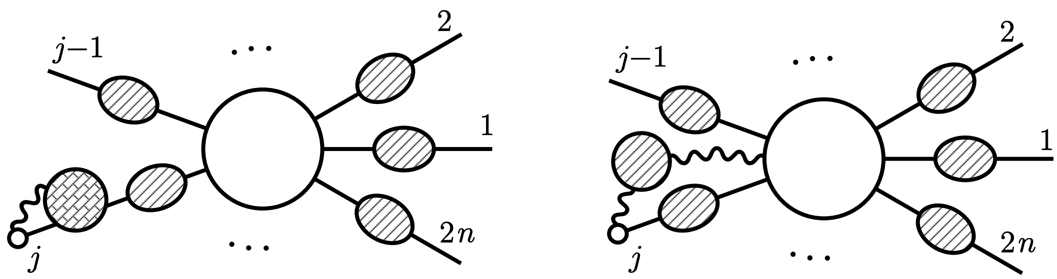

Figure 2: Left: Graphs that do contribute. Right: those do not contribute. Big circles at the center of each graph denote the amputated Green's functions.

near the mass-shell. There is apparently no poles in $\overline{A}(p)$ and $A(p)$ when $\not{p} = m$ so that the right hand side of (3.19) is written as

$$\Delta G^{(2)}(p) \sim \frac{i\Delta z_2}{\not{p} - m} \tag{3.22}$$

implying that the gauge difference reads as the change of wave function renormalization constant z_2:

$$z_2' = z_2 + \Delta z_2 \ . \tag{3.23}$$

Due to $(\not{p}_i - m)$ or $(\not{k}_i - m)$ in (3.16), the surviving part of $\Delta G^{(2n)}$ must have $2n$ one-particle poles. Graphs (see the figures 2), in which the photon in θ (3.5) is attached to its original electron line, that is, graphs including $\overline{A}(p)$ and $A(p)$, have the same pole structure as $G^{(2n)}$ and do contribute, but those, in which the photon goes somewhere other than its original electron line change the pole structure then do not contribute to (3.24). Therefore write the former as $\Delta\overline{G}^{(2n)}$ to find

$$\begin{aligned}
&\Delta\overline{G}^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) \\
&= \sum_{j=1}^{n} \left(\overline{A}(p_j)\Big|_{\not{p}_j = m} G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) - G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) A(k_j)\Big|_{\not{k}_j = m} \right) \\
&= \sum_{j=1}^{n} \frac{\Delta z_2}{z_2} G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) = \frac{n\Delta z_2}{z_2} G^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) \ .
\end{aligned} \tag{3.24}$$

The difference of the gauge dependent part, S_g (3.16), is given by

$$\begin{aligned}
\Delta S_g &\equiv S_g' - S_g \\
&= \prod_{j=1}^{n} \frac{\overline{u}(\boldsymbol{p}_j, s_j')}{\sqrt{(2\pi)^3 2(p_j)_0}} \frac{\not{p}_j - m}{i\sqrt{z_2}} \Delta\overline{G}^{(2n)}(p_1, \cdots, p_n; k_1, \cdots, k_n) \prod_{j=1}^{n} \frac{\not{k}_j - m}{i\sqrt{z_2}} \frac{u(\boldsymbol{k}_j, s_j)}{\sqrt{(2\pi)^3 2(k_j)_0}} \\
&\quad + \Delta z_2 \frac{\partial}{\partial z_2} z_2^{-n} \cdot (z_2^n S_g)
\end{aligned} \tag{3.25}$$

where the first term comes from the change of $G^{(2n)}$ and the second term comes from the change of z_2 (3.23). Due to (3.24), it is apparent that ΔS_g vanishes. There is no gauge dependence in the S-matrix.

4 Functional Method as an Efficient Tool for Handling Gauge Theories

In this section we make a detailed discussion why we can perform a gauge transformation so easily and intuitively in the functional representation. As was mentioned in the introduction, gauge transformation in the canonical operator formalism cannot be allowed at all. In this sense, it is instructive to study the $A_0 = 0$ gauge in the conventional treatment[13]; since there needs a supplementary condition, so called a physical state condition,

implying that *physical state must be gauge invariant.* The statement is apparently contradict with the above situation.

In $A_0 = 0$ gauge, all three components $\boldsymbol{A}$ are assumed dynamical and obey the commutation relations,

$$[\hat{A}_j(\boldsymbol{x}), \hat{\Pi}_k(\boldsymbol{y})] = i\delta_{jk}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad [\hat{A}_j(\boldsymbol{x}), \hat{A}_k(\boldsymbol{y})] = 0 = [\hat{\Pi}_j(\boldsymbol{x}), \hat{\Pi}_k(\boldsymbol{y})]; \quad (j,k=1,2,3) \ . \tag{4.1}$$

Again the caret designates operators. The physical state condition is given as

$$\hat{\Phi}(\boldsymbol{x})|\text{phys}\rangle \equiv \left[\sum_{k=1}^{3}\big(\partial_k \hat{\boldsymbol{\Pi}}_k(\boldsymbol{x})\big) + J_0(\boldsymbol{x})\right]|\text{phys}\rangle = 0 \ , \tag{4.2}$$

where $J_\mu(x)$ is supposed as a c-number current. First this should be read such that *there is no gauge transformation in the physical space*: $\hat{\Phi}$ is the generator of the gauge transformation. However, the representation cannot be obtained within the usual Fock space; since $\hat{\Phi}(\boldsymbol{x})$ is a local operator to result in $\hat{\Phi}(\boldsymbol{x}) = 0$[14], which is another consequence of the above statement. Nevertheless, the state can be expressed in the functional (Schrödinger) representation[15]:

$$\begin{aligned}\widehat{\boldsymbol{A}}(\boldsymbol{x})|\{\boldsymbol{A}\}\rangle &= \boldsymbol{A}(\boldsymbol{x})|\{\boldsymbol{A}\}\rangle \ , \quad \widehat{\boldsymbol{\Pi}}(\boldsymbol{x})|\{\boldsymbol{\Pi}\}\rangle = \boldsymbol{\Pi}(\boldsymbol{x})|\{\boldsymbol{\Pi}\}\rangle \ , \\ \langle\{\boldsymbol{A}\}|\widehat{\boldsymbol{\Pi}}(\boldsymbol{x}) &= -i\frac{\delta}{\delta \boldsymbol{A}(\boldsymbol{x})}\langle\{\boldsymbol{A}\}| \ , \quad \ldots\end{aligned} \tag{4.3}$$

To see the reason consider the state, $|\{\boldsymbol{A}\}\rangle$, which can be constructed in terms of the Fock states as follows: the creation and annihilation operators are given by

$$\begin{aligned}&\widehat{\boldsymbol{A}}(\boldsymbol{x}) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}\sqrt{2|\boldsymbol{k}|}}\big(\mathbf{a}(\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \mathbf{a}^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\big) \\ &[a_i(\boldsymbol{k}), a_j^\dagger(\boldsymbol{k}')] = \delta_{ij}\delta(\boldsymbol{k}-\boldsymbol{k}'), \quad [a_i(\boldsymbol{k}), a_j(\boldsymbol{k}')] = 0 \ ,\end{aligned} \tag{4.4}$$

and the vacuum $|0\rangle$ obeys

$$\mathbf{a}(\boldsymbol{k})|0\rangle = 0 \ . \tag{4.5}$$

Now recall the quantum mechanical case[16]:

$$\begin{aligned}&\hat{q}|q\rangle = q|q\rangle \ , \qquad \hat{p}|p\rangle = p|p\rangle \ , \\ &\hat{q} = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \ , \quad \hat{p} = \frac{1}{\sqrt{2}i}\left(a - a^\dagger\right) \ ; \qquad a|0\rangle = 0 \ ,\end{aligned} \tag{4.6}$$

then

$$\begin{aligned}|q\rangle &= \frac{1}{\pi^{1/4}}\exp\left(-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{(a^\dagger)^2}{2}\right)|0\rangle \ , \\ |p\rangle &= \frac{1}{\pi^{1/4}}\exp\left(-\frac{p^2}{2} + \sqrt{2}ipa^\dagger + \frac{(a^\dagger)^2}{2}\right)|0\rangle \ .\end{aligned} \tag{4.7}$$

These bring us to the expression:

$$\begin{aligned}|\{\boldsymbol{A}\}\rangle \ \simeq \ &\exp\Big[-\frac{1}{2}\int d^3\boldsymbol{x}\ d^3\boldsymbol{y}\ \boldsymbol{A}(\boldsymbol{x})K(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{A}(\boldsymbol{y}) \\ &+ \int d^3\boldsymbol{x}\int d^3\boldsymbol{k}\ \sqrt{\frac{2|\boldsymbol{k}|}{(2\pi)^3}}\ \boldsymbol{A}(\boldsymbol{x})\cdot\mathbf{a}^\dagger(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - \frac{1}{2}\int d^3\boldsymbol{k}\ \mathbf{a}^\dagger(\boldsymbol{k})\cdot\mathbf{a}^\dagger(-\boldsymbol{k})\Big]|0\rangle \ ,\end{aligned} \tag{4.8}$$

where

$$K(\boldsymbol{x}) \equiv \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}|\boldsymbol{k}|e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \ , \tag{4.9}$$

which is apparently divergent:

$$K(\boldsymbol{x}) = O\Big(\frac{\Lambda^2}{|\boldsymbol{x}|}\Big) , \tag{4.10}$$

where Λ is some cut-off. The physical state in the functional representation is thus found as

$$\langle \{\boldsymbol{A}\}|\hat{\Phi}(\boldsymbol{x})|\text{phys}\rangle = \left(-i\boldsymbol{\nabla}\frac{\delta}{\delta \boldsymbol{A}(\boldsymbol{x})} - J_0(\boldsymbol{x})\right)\Psi_{\text{phys}}[\boldsymbol{A}] = 0 , \tag{4.11}$$

where

$$\Psi_{\text{phys}}[\boldsymbol{A}] \equiv \langle \{\boldsymbol{A}\}|\text{phys}\rangle . \tag{4.12}$$

Therefore physical state can be obtained under the functional representation, implying that gauge transformations are permissible. Now we should see the reason: within a single Fock state the physical state condition (4.2) merely implies $\hat{\Phi}(\boldsymbol{x}) = 0$. However, we should bear the following fact in mind: *the functional representation consists of infinitely many collections of inequivalent Fock spaces*; since the inner product of $|\{\boldsymbol{A}\}\rangle$ (4.8) to the Fock vacuum is found to be

$$\begin{aligned} \langle \{\boldsymbol{A}\}|0\rangle \ &\sim\ \exp\left[-\frac{1}{2}\int d^3\boldsymbol{x}\ d^3\boldsymbol{y}\ \boldsymbol{A}(\boldsymbol{x})\ K(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{A}(\boldsymbol{y})\right] \\ &\sim\ \exp\left[-\frac{\Lambda^2}{2}\int d^3\boldsymbol{x} d^3\boldsymbol{y}\frac{\boldsymbol{A}(\boldsymbol{x})\boldsymbol{A}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\right] \stackrel{\Lambda\to\infty}{\longrightarrow} 0 \ . \end{aligned} \tag{4.13}$$

Note that in the functional space $\boldsymbol{A}(\boldsymbol{x})$ is far from a Fourier transformable form[6]. Therefore the space of $\boldsymbol{A}(\boldsymbol{x})$ which makes the exponent in the last relation finite is almost measure zero. Then we can say that (4.13) happens in *any value of* $\boldsymbol{A}(\boldsymbol{x})$. Thus the functional representation for *any* $\boldsymbol{A}(\boldsymbol{x})$ is orthogonal to the Fock state, that is, inequivalent to the Fock state. Any local first class constraint, (apart from the mathematical rigorousness of that), can be realized by means of the functional representation.

This fact that the functional representation contains an infinite set of the Fock states enables us to perform an explicit gauge transformation and prove gauge independence without recourse to any physical state conditions in path integral. (Recall that (2.27) and (2.28) are the essential ingredients toward the path integral representation.)

5 Discussion

In this paper, we have built up the path integral formula of the abelian gauge theory in which an arbitrary gauge function is included. Although in the operator formalism the support of the function must be in a spacelike region, thus generality is lost, there is no restriction in the Euclidean path integral expression so as to be able to move, for instance, from the Coulomb to the Landau gauge. By utilizing arbitrariness of the function, gauge independence of the free energy is quickly understood (but that of the S-matrix needs further considerations.) Furthermore, a closer inspection reveals the reason why gauge transformations are so easily managed in the path integral.

As was seen in the discussion of the S-matrix, multiplying wave functions, that is, the on-shell condition, is indispensable for the proof of gauge independence. The on-shell condition belongs to one of the physical state conditions. Hence, in scattering theories or in perturbation theories, usual (LSZ-)asymptotic states[18], (3.15), are known to behave as the physical states. However, it is not so easy to find out the form of the physical state in a nonperturbative manner. Steinmann's first intention seems to explore this: indeed, the physical electron (1.2) by taking ϕ_μ as the Coulomb case (1.12),

$$\Psi^{\text{D}}(x) = \exp\left[-ie\frac{\boldsymbol{\nabla}\cdot A}{\boldsymbol{\nabla}^2}\right]\psi(x) , \tag{5.1}$$

is the one introduced by Dirac[17], which is locally gauge invariant as well as globally charged. The existence of such a state implies an evidence of electron as a real particle[19].

[6]Recall that even the free theory the action is divergent, implying those do not belong to $\mathcal{L}^2$ class.

The issue should then be raised to the nonabelian gauge case (QCD). In order to study dynamics of the quark confinement, it is important to examine whether physical charged state can be constructed or not. The key to this direction would be to notice the Gribov ambiguity[20]: in a smaller region, the Coulomb gauge is well-defined, that is, no gauge degrees of freedom being left owing to the asymptotic freedom. The larger a region, however, the more nontrivial degree comes into a part[21]. Since gauge invariance is essential to comprehend the quark confinement, the path integral must be useful. Therefore in order for the theory to be well-defined in terms of the path integral the integration region of the gauge fields must pertain to that of the Lagrangian, which would finally give us a compact integration of gauge fields given by the lattice QCD[22]. A work in this direction is in progress.

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